Encouraging Mathematical Thinking ~with Power & Simplicity

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Encouraging Mathematical Thinking ~with Power & Simplicity

My interest is in the *long-term growth of mathematical thinking* from new-born child to adult.

To understand *how we think mathematically* and *how we can help students learn to think mathematically.*

Encouraging Mathematical Thinking ~with Power & Simplicity

Major ideas:

Learning *procedures* enables us to *do* maths, but perhaps not to *think* about it.

Over the long-term, powerful mathematical thinking requires ideas to be *compressed* from procedures to *do* mathematics (such as counting) into *thinkable concepts* (such as number).

Encouraging Mathematical Thinking ~with Power & Simplicity

Major ideas:

Thinkable concepts require two aspects:

1. **focus on relevant aspects** so that complicated information is linked together in ways that are simple,
2. **pay attention to negative aspects**, (eg ‘take away leaves less’) that were once successful (every day experience) but later fail ($5-(-3) = 8$).

Encouraging Mathematical Thinking ~with Power & Simplicity

Major ideas:

Teachers need to act as *mentors* to help students to:

1. ***compress knowledge* into thinkable concepts relevant for new situations,**
2. ***rethink older experiences* that cause confusion in new situations.**

Embodiment and Symbolism

In school mathematics there are two parallel developments of mathematical thinking:

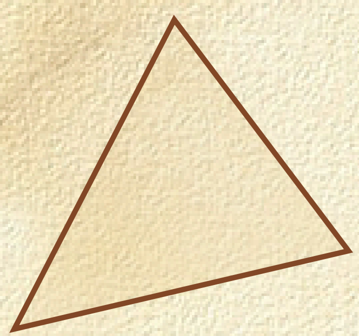
embodiment (human perception & action)

symbolism (to represent procedures (eg counting) as thinkable concepts (number)

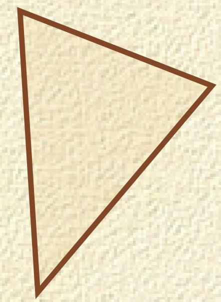
In later development there is a third:

formal mathematics using axioms and formal proof.

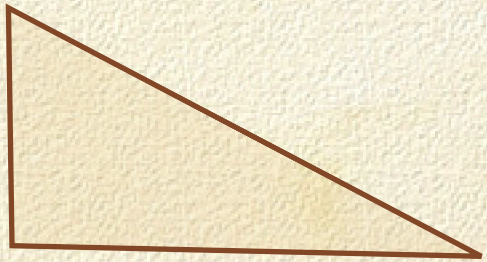
The Embodied World



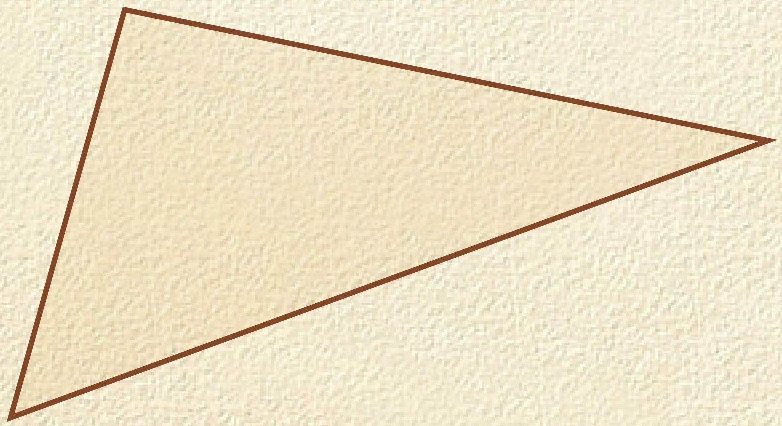
a triangle



another triangle



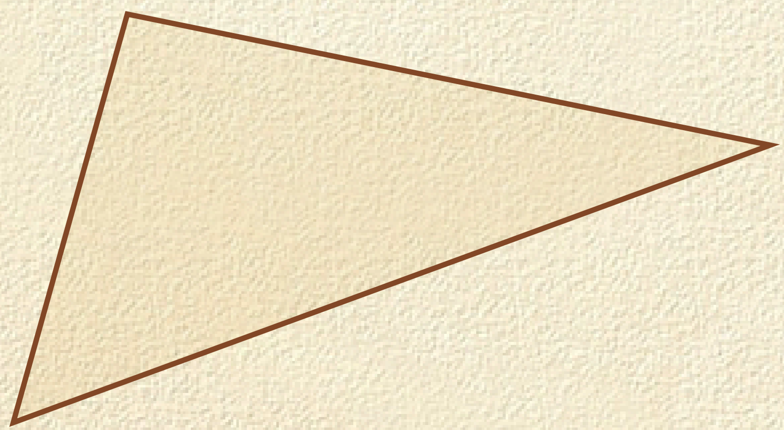
another triangle



in our mind we can imagine any three-sided figure as a 'triangle'

The Embodied World

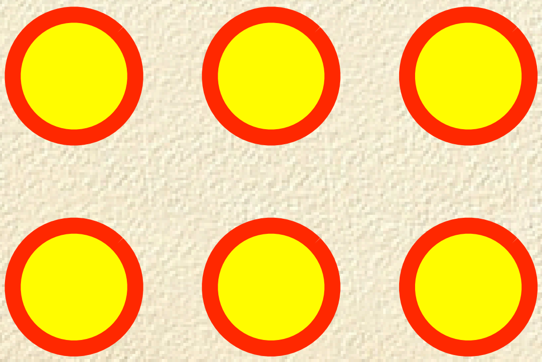
a ‘triangle’ seen not just as a specific example but as a prototype representing ‘any triangle’



This *compression of knowledge* allows a single triangle to represent *any* triangle.

The Embodied World

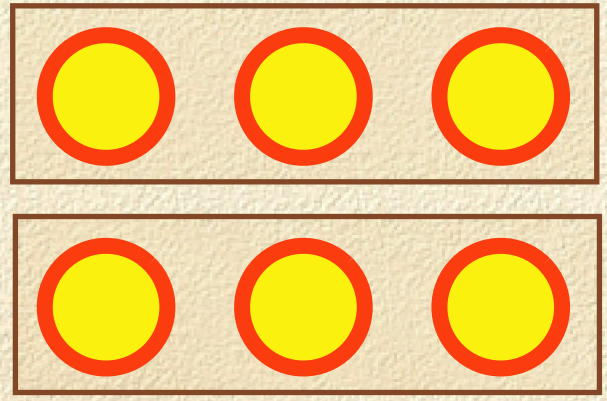
an array seen not just as a specific example 2 x 3 but as a prototype representing *m* x *n*



The Embodied World

We can see that 2 x 3

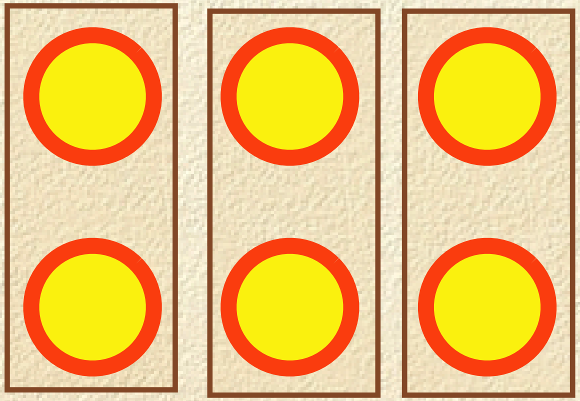
2 sets of 3



The Embodied World

We can see that 2 x 3 is the same as 3 x 2

3 sets of 2

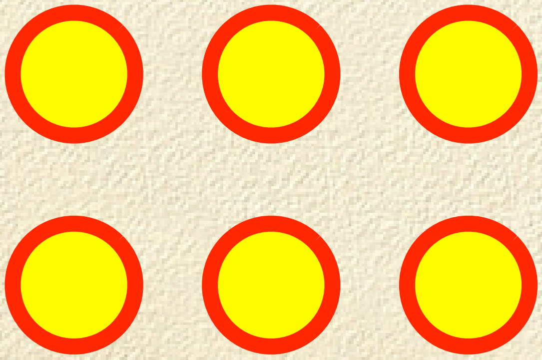


The Embodied World

We can see that 2 x 3 is the same as 3 x 2

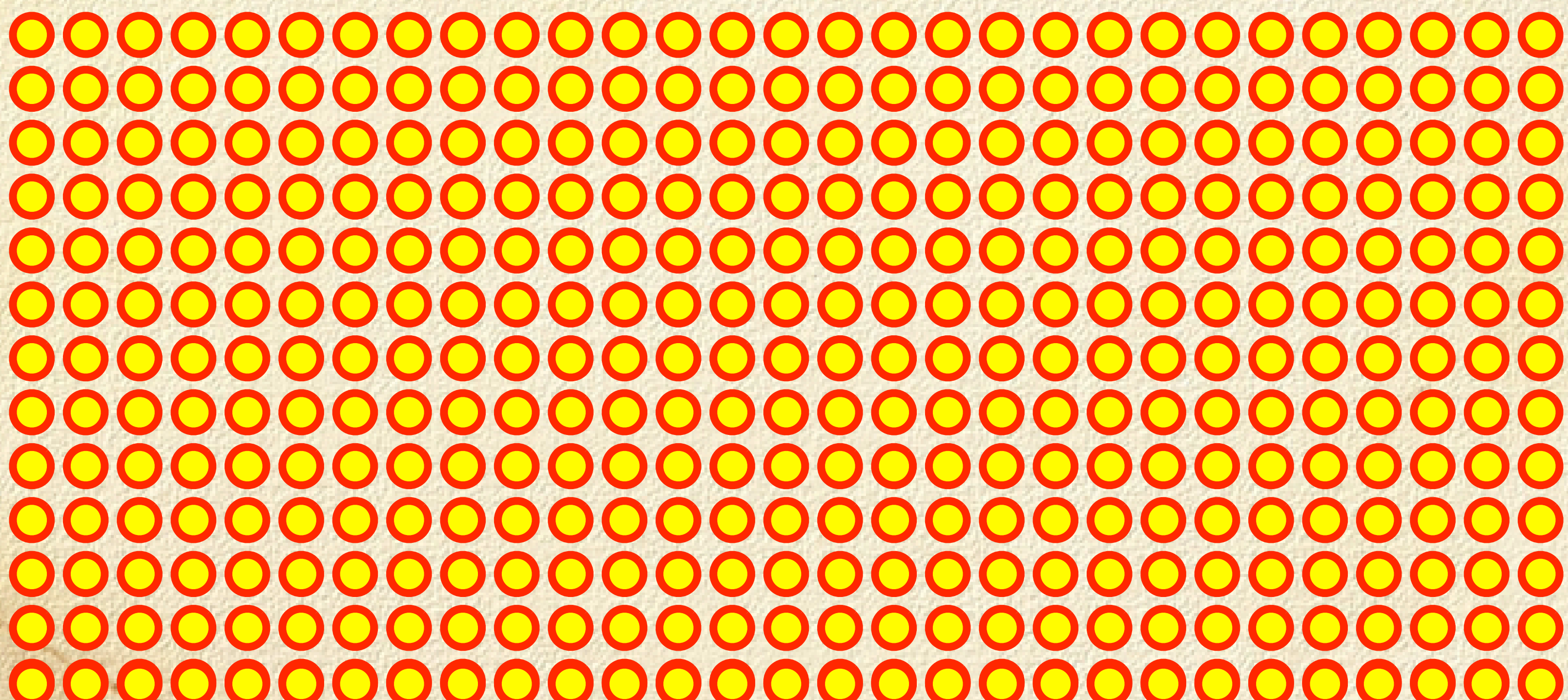
and as a prototype

$$m \times n = n \times m$$



The Embodied World

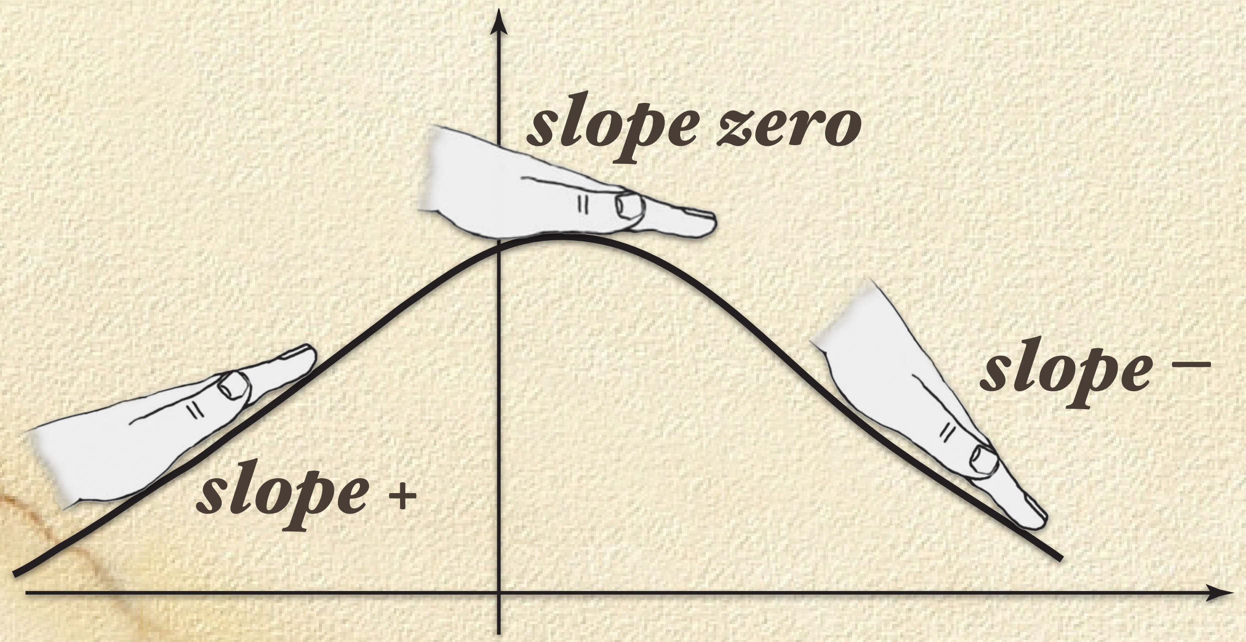
We can see *each row has the same number* (*m*) and *each column the same number* (*n*), so *m* x *n* = *n* x *m* (without counting *m* or *n*)



A picture can represent a prototype

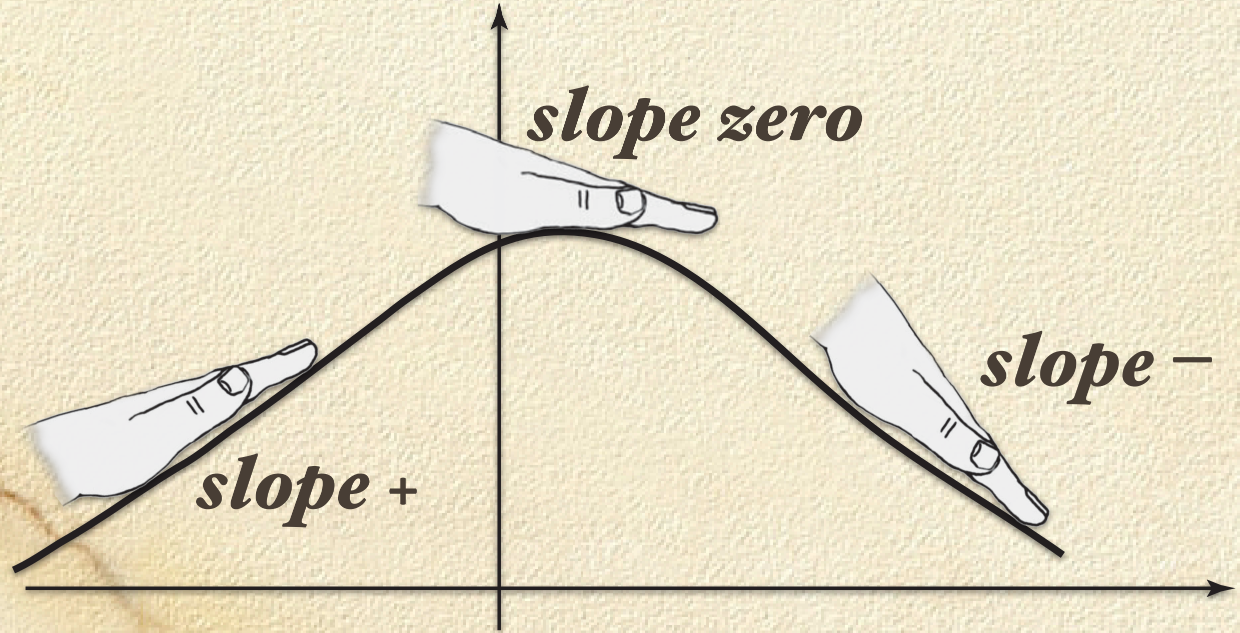
The Embodied World

Embodiment includes such things as moving our hand to draw a curve, sensing how it might slope upwards initially, then lie flat, then slope down, to see a maximum value on a graph.



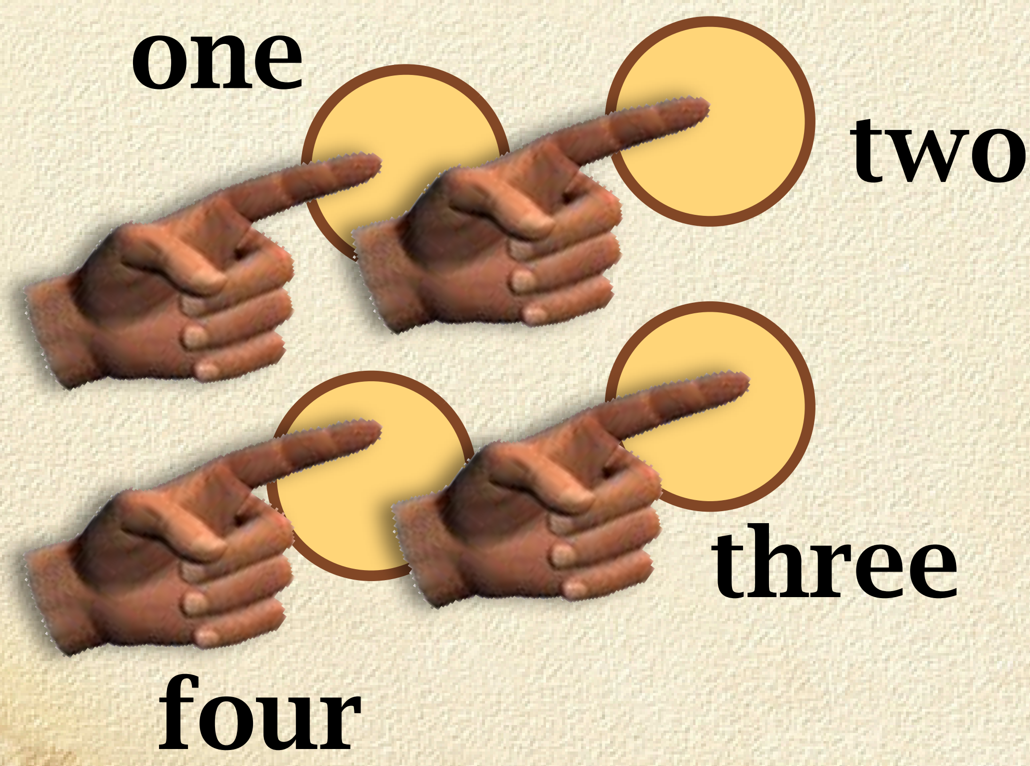
The Embodied World

The world of embodiment operates at all levels of mathematical thinking, underlying mathematical concepts with embodied actions.



The Relationship between Embodiment and Symbolism

The symbolism of arithmetic and algebra develops from performing actions that we repeat until they can be performed automatically, such as counting.



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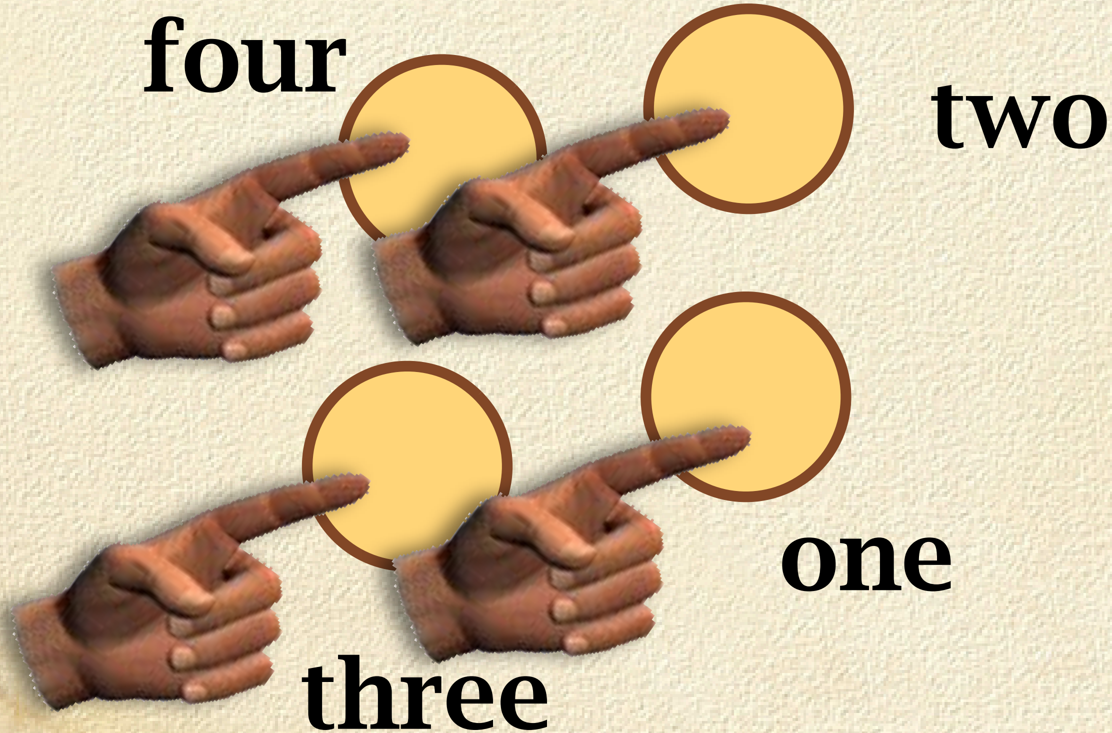
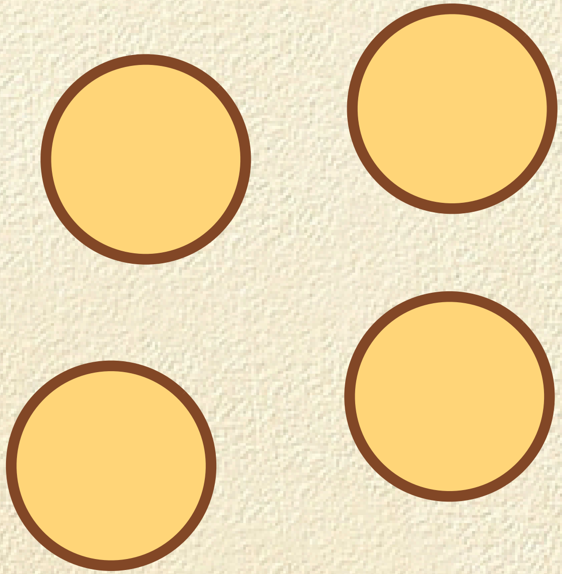


The Relationship between Embodiment and Symbolism

The symbolism of arithmetic and algebra develops from performing actions that we repeat until they can be performed automatically, such as counting.

Each time the effect is the same:

four

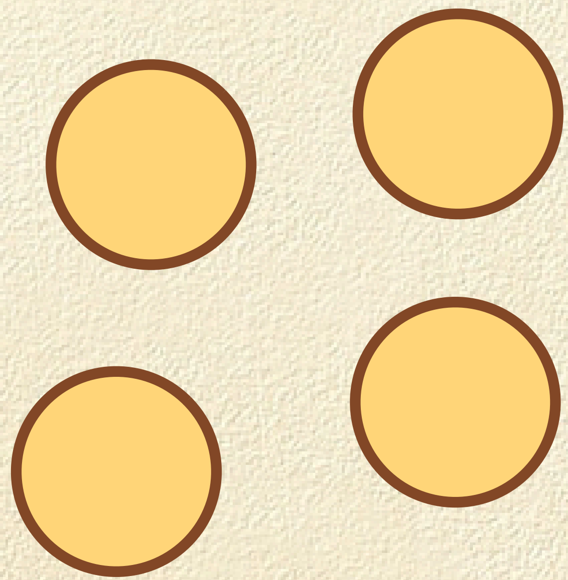


The Relationship between Embodiment and Symbolism

There is a change in our focus of attention from the actions we do (counting) to the effect of the action (number)

Each time the effect is the same:

four



This is *compression* from procedure to concept

Compression of Knowledge: Symbols as process and concept

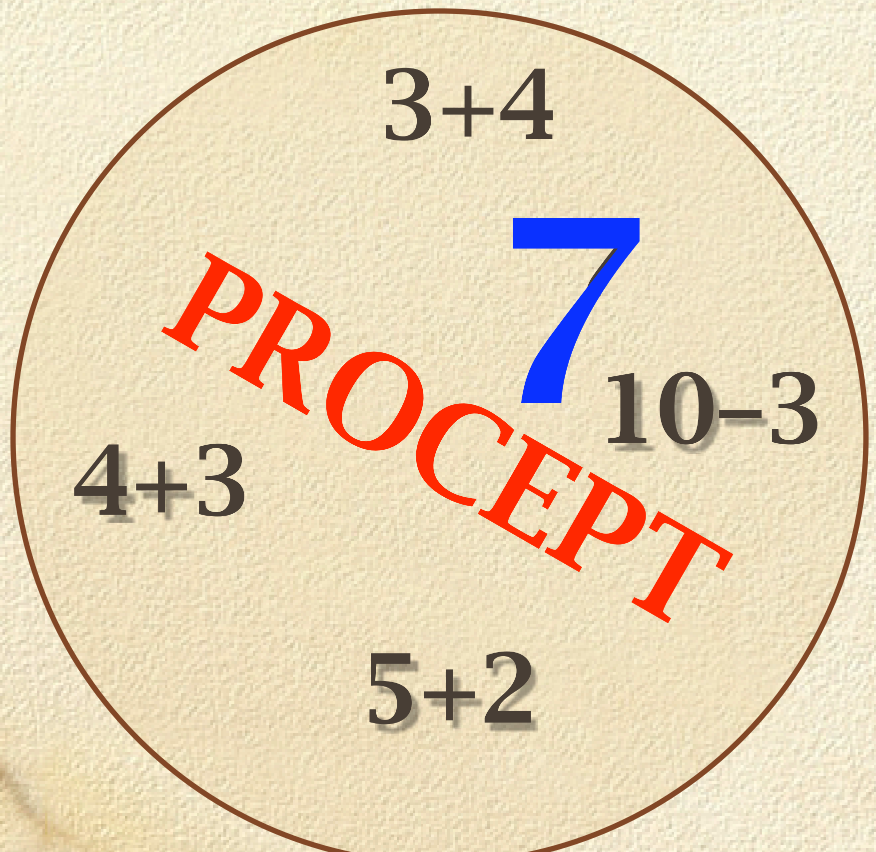
Many symbols in arithmetic, algebra, calculus etc represent both a process e.g.3+4 represents *addition*
and also the *sum* 3+4 is 7.

The *process* of addition is *compressed* into the *concept* of sum.

Compression of Knowledge: Symbols as process and concept

Different symbols can represent different actions *with the same effect*.

compression into a rich thinkable concept



This combination of symbols as **pro**cesses (with the same effect) and con**cept** is called a **procept**.

Compression of Knowledge: Symbols as process and concept

Symbol	Process	Concept
$3+2$	addition	sum
-3	subtract 3	negative 3
$3/4$	division (sharing)	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$\sin(A)$	opposite/hypotenuse	sine function
$f(x)$	evaluation	function f
$\lim \Sigma 1/n^2$	tend to limit	limit value
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
v	translation	vector
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n

Compressing Actions into Symbols

What happens if students do not compress step-by-step procedures into thinkable concepts?

Answer: They can do routine examples but not think flexibly when the examples are a little different.

Compressing Actions into Symbols

Circle the expressions that give the same result:

Write another expression that is the same.

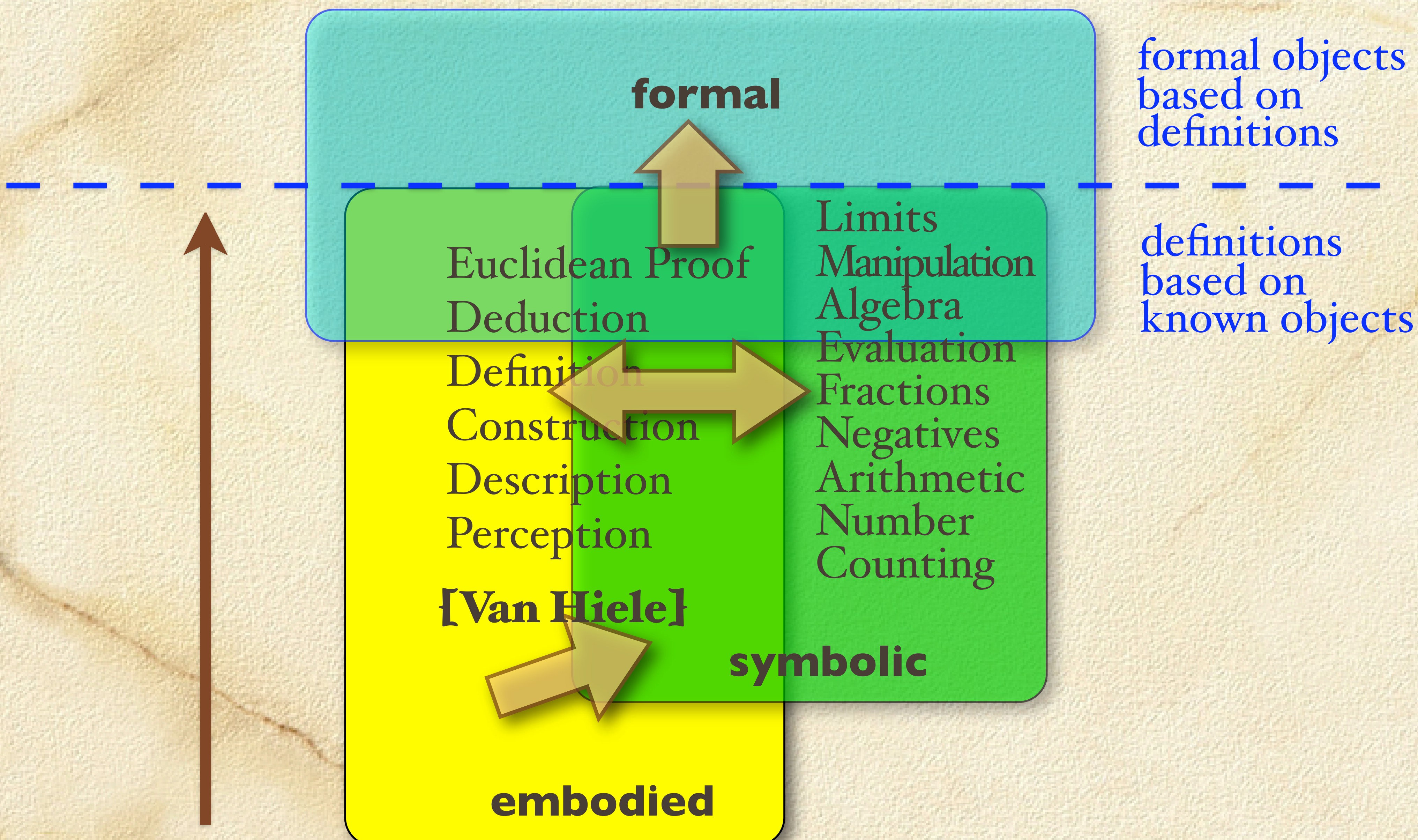
Student "John" wrote:

$(x+3)(x-2)$, x^2+5x-6, x^2+x-6. $(x-2)(x+3)$

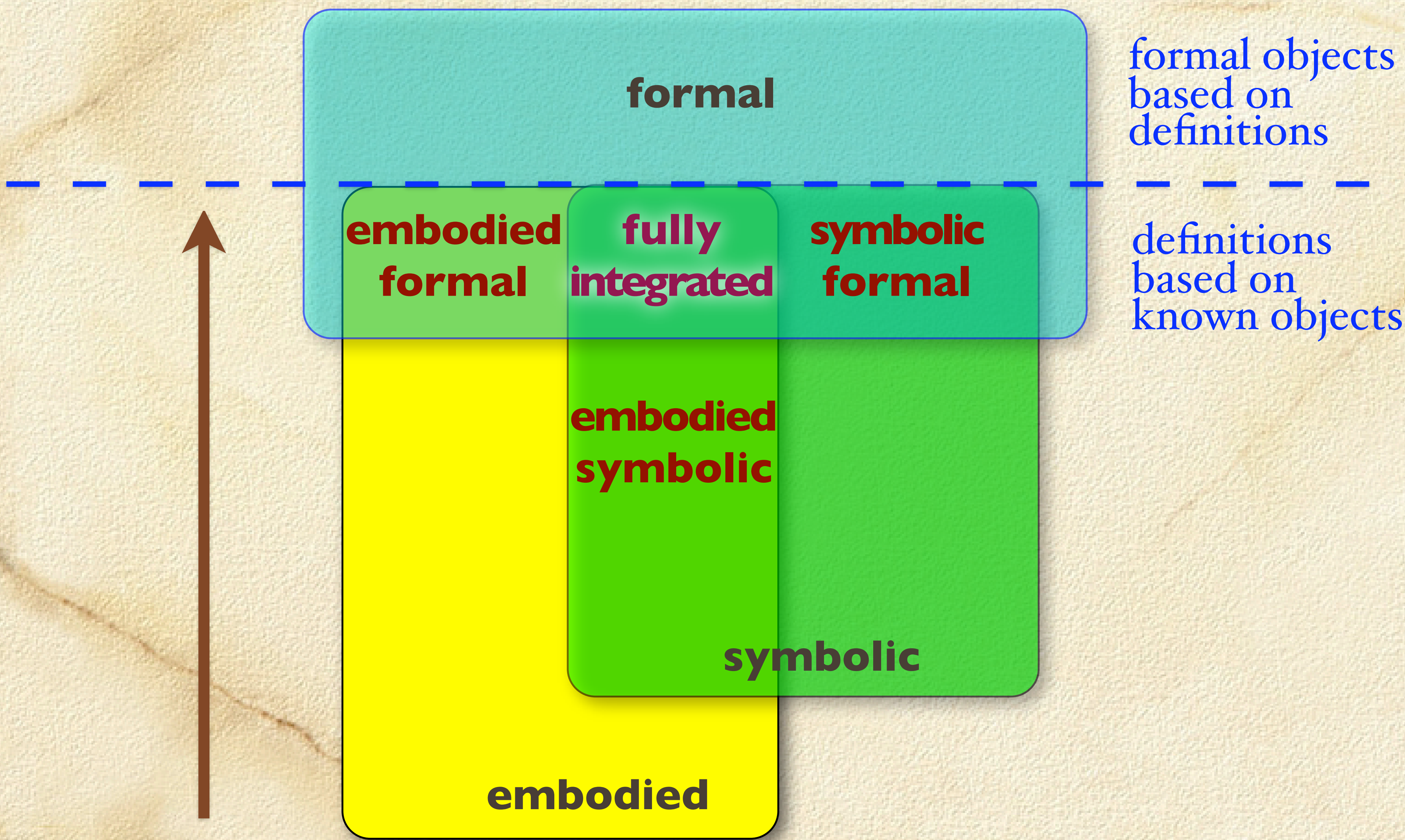
A B C D

However, John factorized C to get D.
He did not realize that A and D were the same.
Later he found calculus too complicated, learnt by rote and made many errors.

Three Worlds of Mathematics



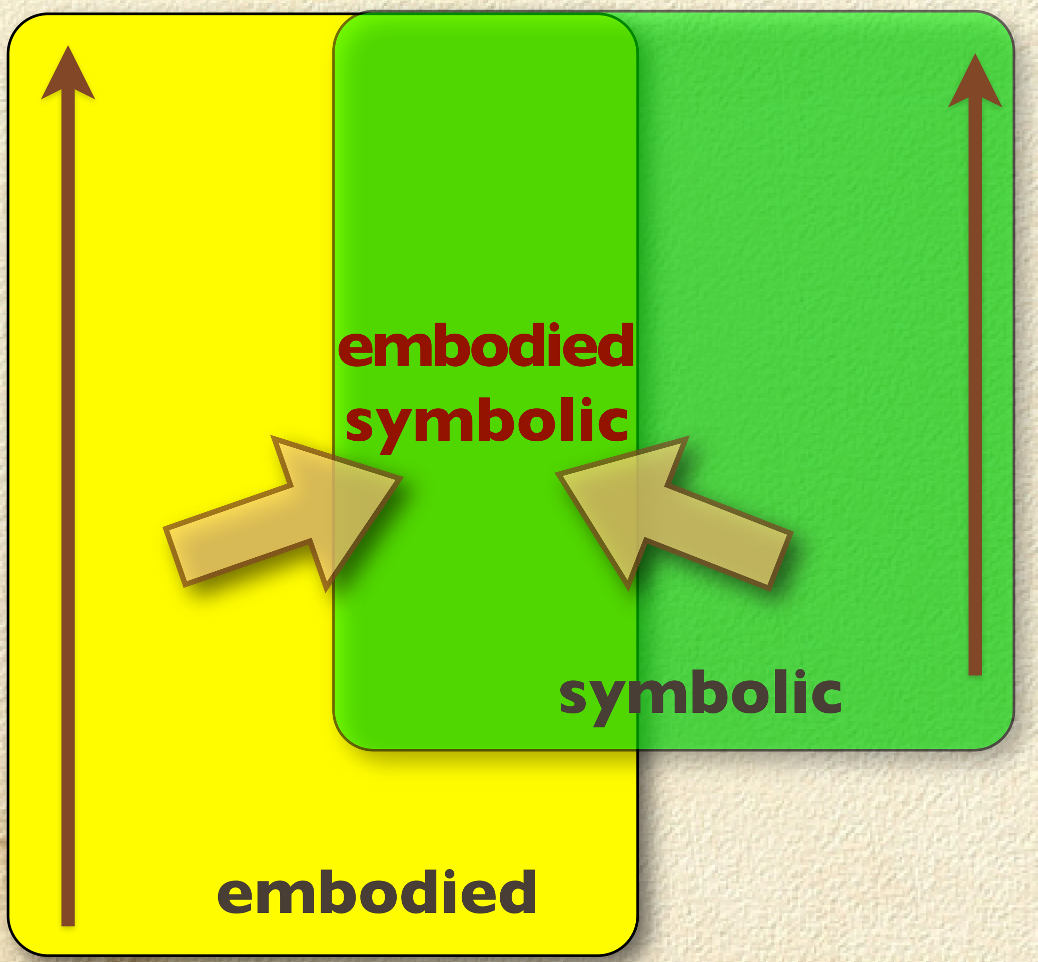
Three Worlds of Mathematics



Embodiment and Symbolism

In school mathematics our interest is mainly on the relationship between embodiment and symbolism

increasingly sophisticated use of **prototypes** and shift to **definitions** as **thinkable concepts**

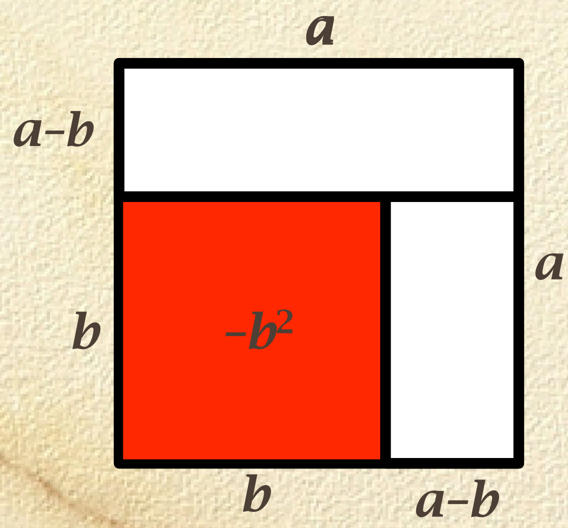


increasingly sophisticated **compression** of procedures into **procepts** as **thinkable concepts**

Can we use Embodiment to give meaning to Symbolism?

Example:

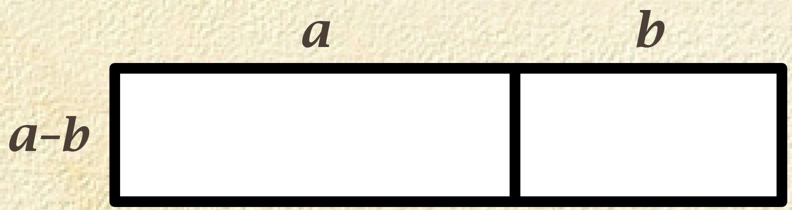
The difference between two squares:



Can we use Embodiment to give meaning to Symbolism?

Example:

The difference between two squares:



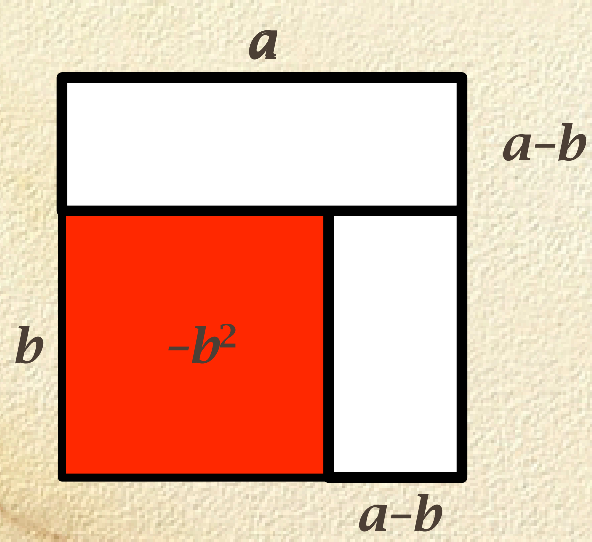
$$a^2 - b^2 = (a-b)\,(a+b)$$

What happens when some of the quantities are negative?

Example:

The difference between two squares:

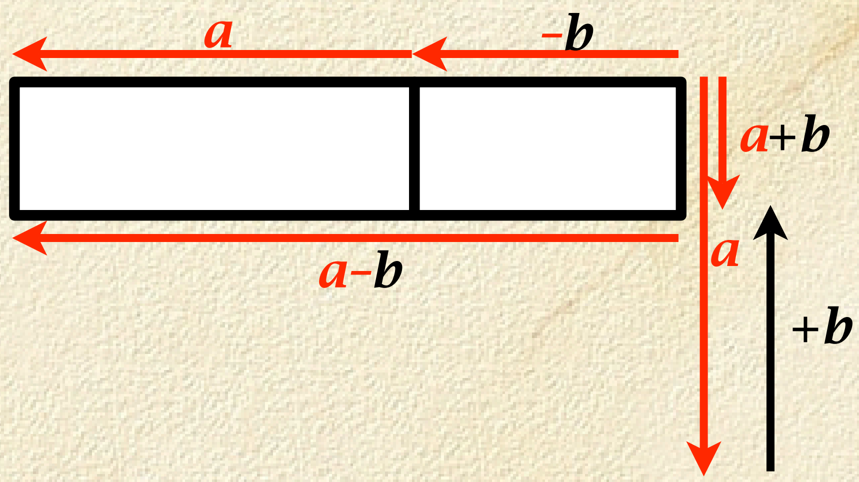
a,*b* positive.



$$a^2-b^2 = (a-b)\,(a+b)$$

a negative

b positive, $|b| < a$.



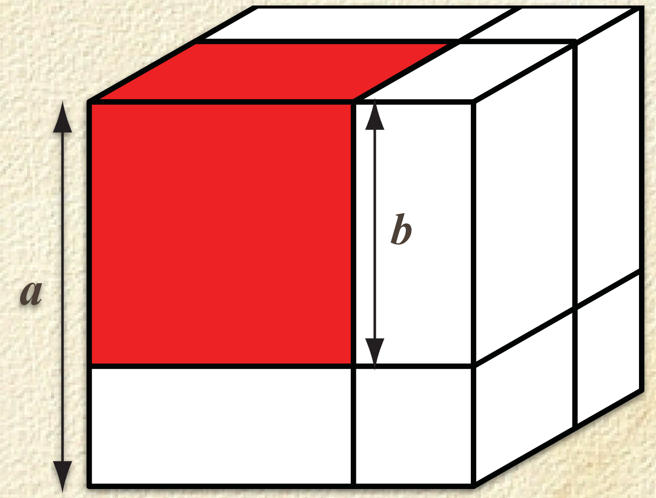
It still works but becomes complicated

What happens when we generalize to higher powers?

If we try to interpret

$a^3 - b^3$

$a^4 - b^4$



$a^3 - b^3$

as pictures in 3 and 4 dimensions,
3 dimensions is possible but more complicated,
4 dimensions takes us beyond our 3D world.

What happens when we generalize to higher powers?

The difference of two squares by manipulation of symbols:

$$(a+b)\,(a-b) = (a+b)a - (a+b)b$$

$$= a^2 + ba - ab - b^2$$

$$= a^2 - b^2$$

What happens when we generalize to higher powers?

The difference between two cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

The difference between two 4th powers uses the difference between two squares:

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$$
$$= (a - b)(a + b)(a^2 + b^2)$$

The symbolism generalizes easily, while the embodiment is more complicated.

What happens when we generalize to higher powers?

The consequence is that, as algebra becomes more sophisticated, it is more important to give meaning to the *symbolism* rather than rely on an embodiment.

Previous experience can cause problems

There is a more serious problem:
We build on ideas we already have.
Some worked well in one context.
e.g. 'taking away something leaves less'
(which is generally true)
is false with negative numbers:

$$5-(-2) = 7.$$

Previous experience can cause problems

A **met-before** is an idea we met earlier which affects the way we think now.

e.g. arithmetic expressions have an answer.

2+3 *is* 5.

But what is $2+3x$?

It has no ‘answer’ unless x is known.

Some students do the arithmetic and leave the x alone to wrongly get $2+3x$ is $5x$.

Previous experience can cause problems

There are many met-befores that cause difficulties in new contexts.
e.g. expressions can be simplified by moving like terms together.

$$2a + 3b + 4a$$

move term over

$$2a + 4a + 3b$$

add like terms

$$6a \quad + 3b$$

Previous experience can cause problems

Moving terms when solving equations is performed procedurally by many students using the rule 'change side, change sign'.

$$4x + \boxed{3} = x + 9$$

$$4x = x + 9 \boxed{-3}$$

change sides change sign

For students who do not understand it is a *procedural embodiment*.
Picking up and moving terms
+ 'magic' (change signs)

Previous experience can cause problems

Some students learning algebra procedurally can later learn to give it meaning.

Many students learning algebra procedurally can solve routine problems but not think about mathematics flexibly.

I suggest that flexible mathematical thinking is very difficult for students who already see arithmetic purely as procedure.

Mathematical thinking needs to be nurtured long-term.

International Examples

At Warwick University we have several studies performed in different countries which show that teaching procedures to pass tests is causing problems.

Students can solve routine problems but not problems that are a little different.

International Examples

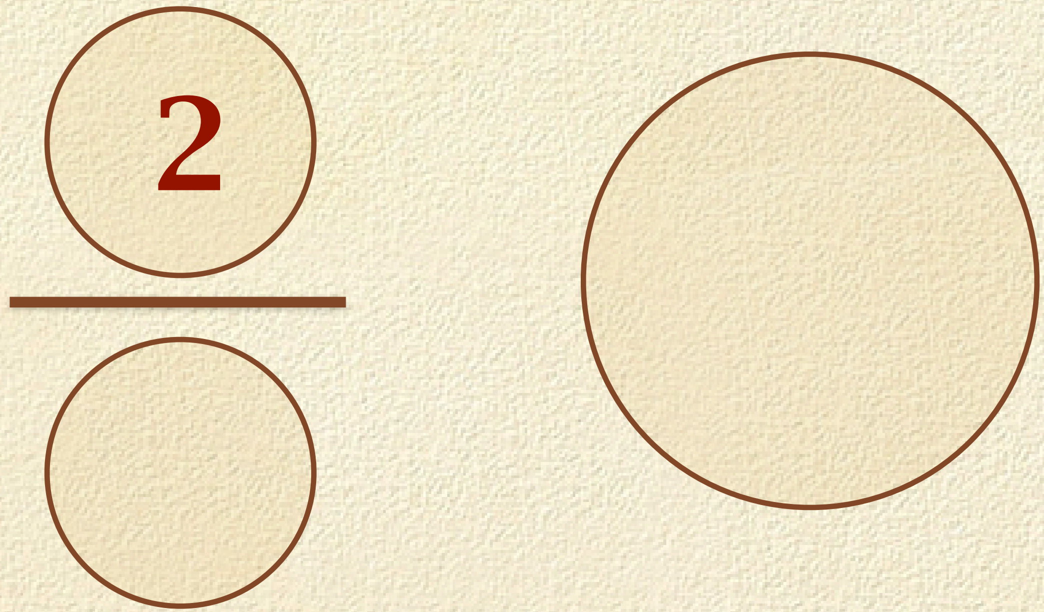
In Malaysia a study of teaching fractions revealed the desire to show flexibility but the teaching remained procedural:

'Flexible' here meant showing different procedures for the same operation.

So 2/5 of 20 could be performed in two different ways:

International Examples

2/5 of 20:



Teacher:

What do we put in the top circle … the nu…

Class: the numerator!

International Examples

2/5 of 20:

$$\frac{2}{5}$$

Teacher:

What do we put in the bottom circle … the de…

Class: the denominator!

International Examples

2/5 of 20:

$$\frac{2}{5} \times 20$$

Teacher:

What does ‘of’ mean? ‘Of’ means mul…

Class: Multiply!

International Examples

2/5 of 20:

$$\frac{2}{5} \times 20$$

And so the problem continues, one method is to divide the 20 by 5 to get 4.

International Examples

2/5 of 20:

$$\frac{2}{\quad} \times 4$$

And so the problem continues, one method is to divide the 20 by 5 to get 4.

Then 2 x 4 is 8.

International Examples

2/5 of 20:

$$\frac{2}{5} \times 20$$

Another method is to multiply the 2 times 20 to get 40.

International Examples

2/5 of 20:

$$\frac{\quad}{5} \times 40$$

Another method is to multiply the 2 times 20 to get 40.

Then divide by 5 to get 8 again.

International Examples

The students get proficient at two different procedures and perform well on routine examples...

But there is no discussion on the idea that the two procedures have the same effect.

There is little compression of knowledge and the notion of fraction remains procedural with little flexible mathematical thinking.

International Examples

In Brazil a group of teachers found their students had difficulty with linear equations, so in quadratic equations, they taught mainly the formula.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to give the solutions of $ax^2+bx+c = 0$.

International Examples

The students are given the following problem:

A student John says the solutions of $(x-2).(x-3) = 0$ are 2 and 3.
Is John right? Discuss.

None of the students substituted $x=2$, $x=3$ to see if they were solutions.
Some tried to use the formula.
Only 6 out of 70 were correct.

International Examples

In Turkey, the curriculum is based on the function concept.

One teacher Ahmet focused on the simplicity of the concept and built knowledge from that.

One teacher Burak knew the students had difficulties and focused on the procedures needed to pass the exam.

International Examples

Ahmet:

A function is a very simple idea:

You have two sets A and B and for each element *x* in A there is precisely one corresponding element *y* in B, called *f*(*x*).

That's it!

He gave many examples, focusing on how they all satisfy the simple idea and showed how the 'vertical line test' came from this idea.

International Examples

Ahmet:

He explained that the constant function, where every x in A maps onto the same element c in B is the *simplest* case. There is nothing to do except link every x in A to the one element c.

International Examples

Burak:

knew students had difficulties with functions. He gave the definition then explained the vertical line test as a way of checking for a function, practicing with many examples.

He knew the students had difficulty with the constant function because $f(x)$ did not vary with x.

So he explicitly *taught* that the formula for $f(x)$ did not need to contain x.

International Examples

Dealing with the inverse of a function:

Ahmet discussed when the correspondence would work in reverse, bringing out the need for the function to be *onto* (to have every y in B having a corresponding element in A) and to be *one-to-one* (so that there was only one possible corresponding element).

Having established the simple idea of inverse, he then considered many examples in algebra, trigonometry etc, patiently referring to the simple idea they had discussed.

International Examples

Dealing with the inverse of a function:

Burak knew students had difficulties, so he chose to start with an example, the inverse of

$$y = 2x+3$$

He explained the steps, first to subtract 3 to get

$$y - 3 = 2x$$

divide by 2: $x = (y - 3)/2$

interchange x, y**:** $y = (x - 3)/2$

to get the inverse.

He then practiced many examples.

International Examples

Burak's students could solve routine problems they had been taught but they made more errors and lacked the flexible mathematical thinking shown by the students of Ahmet.

Reflections

Meaning starts with practical embodiment but needs to shift to meaningful symbolism.

Teaching procedures is important but flexible mathematical thinking requires *compression of knowledge* into *thinkable concepts* that can be easily manipulated in the mind.

It also requires *resolution of conflicts* arising from earlier knowledge that worked then, but does not work in the new situation.

Reflections

Long-term curriculum design must take into account *how* individual students learn, building from embodiment to compression of symbolic procedures into *thinkable concepts* that can be manipulated in the mind.

The teacher should act as *mentor* to encourage the construction of thinkable concepts and to dicuss earlier experiences (*met-befores*) involving ideas that worked then, but require rethinking to work in a new situation .

David Tall's Home Page

http://www.davidtall.com/

David Tall

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Personal Pages

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Musical reunion

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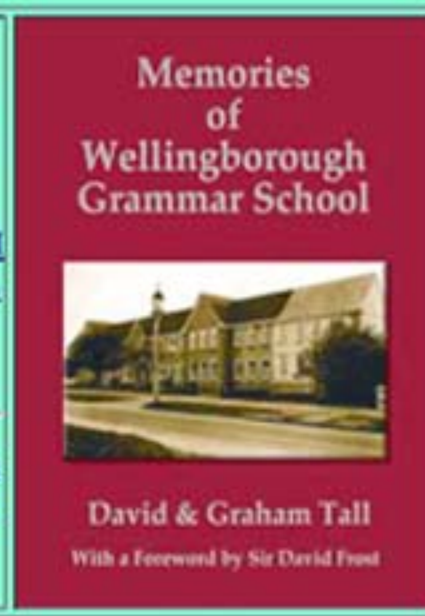
Newspaper report from the *Evening Telegraph*



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David Tall - Professor in Mathematical Thinking

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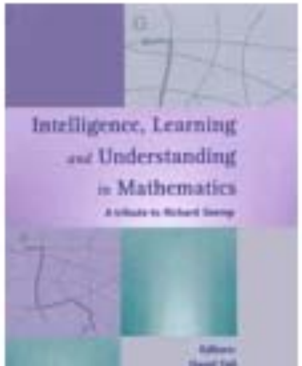

David retired in September 2006 and is now Emeritus Professor of Mathematical Thinking at Warwick
His retirement conference RETIREMENT AS PROCESS AND CONCEPT, shared with Eddie Gray at Charles University Prague, is available for download and pictures of the conference are here.